The EUCLID ALGORITHM is “TOTALLY” GAUSSIAN

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Study of Local Limit Theorems, with their speed of convergence.

Less studied than Central Limit Theorems,

even in the simplest probabilistic framework.

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II- Distributional results which are already known

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Local limit theorems in the particular case of a lattice cost.

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$$\left\{\begin{array}{ccccccc} u_0 & = & m_1 u_1 & + & u_2 & & 0 < u_2 < u_1 \\ u_1 & = & m_2 u_2 & + & u_3 & & 0 < u_3 < u_2 \\ \dots & = & \dots & + & & & \\ u_{p-2} & = & m_{p-1} u_{p-1} & + & u_p & & 0 < u_p < u_{p-1} \\ u_{p-1} & = & m_p u_p & + & 0 & & u_{p+1} = 0 \end{array}\right\}$$

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CFE of $\frac{u}{v}$:

$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p}}}},$$

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– the $\gcd(u, v)$ itself

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– the Continued Fraction Expansion CFE (u/v)

Often used directly in computation over rationals.

The main object of interest here.

A basic algorithm ... Perhaps the fifth main operation?

The main costs of interest for the continued fraction expansion

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if $d = 1$, then $\widehat{D}$:= the number of iterations

if $d = \mathbf{1}_{m_0}$, then $\widehat{D}$:= the number of digits equal to m_0

if $d = \ell$ (the binary length), then $\widehat{D}$:= the length of the CFE

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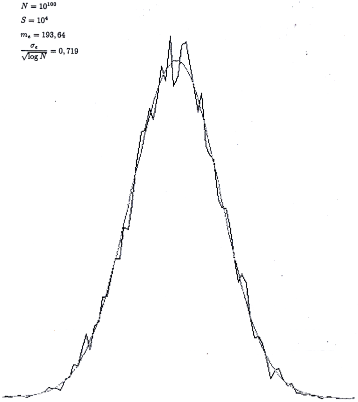
However, it is also interesting to study general digit costs,

They give rise to various observables on the Continued Fraction expansion

For instance $d(m) = \log m$, ... related to the Khinchine constant.

Main probabilistic questions on the Continued Fraction Expansion
... and its "total" cost $\widehat{D}$



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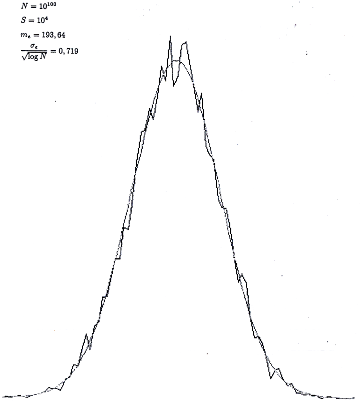

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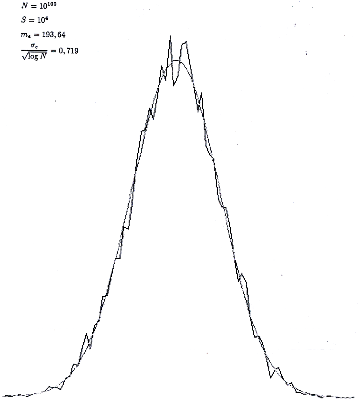

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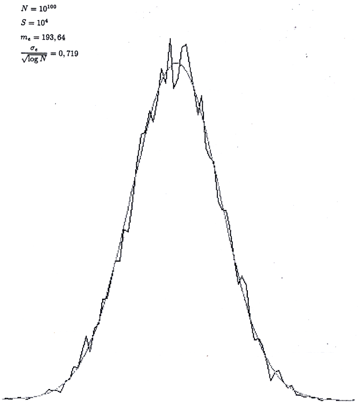

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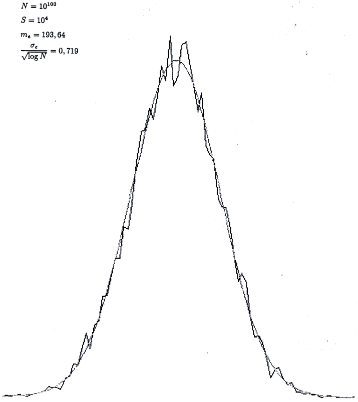

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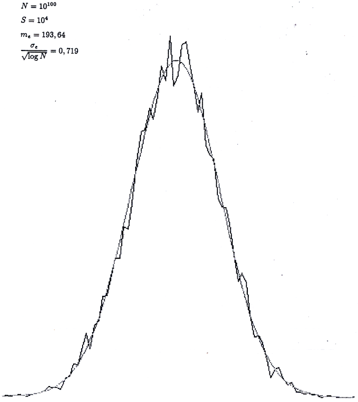

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Which speed of convergence?

The underlying dynamical system (I).

The trace of the execution of the Euclid Algorithm on (u_1, u_0) is:

$$(u_1, u_0) \to (u_2, u_1) \to (u_3, u_2) \to \ldots \to (u_{p-1}, u_p) \to (u_{p+1}, u_p) = (0, u_p)$$

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Replace the integer pair (u_i, u_{i-1}) by the rational $x_i := \dfrac{u_i}{u_{i-1}}$.

The division $u_{i-1} = m_i u_i + u_{i+1}$ is then written as

$$x_{i+1} = \frac{1}{x_i} - \left\lfloor \frac{1}{x_i} \right\rfloor \qquad \text{or} \qquad x_{i+1} = T(x_i), \qquad \text{where}$$

$$T : [0,1] \longrightarrow [0,1], \quad T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for} \ \ x \neq 0, \quad T(0) = 0$$

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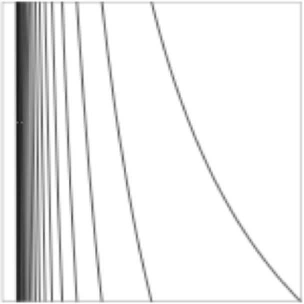
An execution of the Euclidean Algorithm $(x, T(x), T^2(x), \ldots, 0)$

= A rational trajectory of the Dynamical System $([0,1], T)$

= a trajectory that reaches 0.

The dynamical system is a continuous extension of the algorithm.



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$$T_{[m]} : \left]\frac{1}{m+1}, \frac{1}{m}\right[\longrightarrow]0,1[,$$

$$T_{[m]}(x) := \frac{1}{x} - m$$

$$h_{[m]} :]0,1[\longrightarrow \left]\frac{1}{m+1}, \frac{1}{m}\right[$$

$$h_{[m]}(x) := \frac{1}{m+x}$$

$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p}}}} = h_{[m_1]} \circ h_{[m_2]} \circ \ldots \circ h_{[m_p]}(0)$$

The discrete algorithm is extended into a continuous process.

Two types of weighted trajectories and two probabilistic models:

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First model : Study of truncated real trajectories "at depth n"

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on $\Omega_N := \{x = u/v \in \mathcal{I}, v = N\}$

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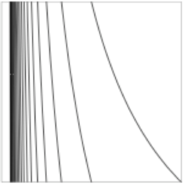
We wish to compare these two "observables" .

Since the discrete data are of zero measure amongst the continuous data,

we need a "transfer from continuous to discrete".

A main tool in both probabilistic models: The transfer operator.

The transfer operator



$\mathcal{H} :=$ the set of the inverse branches of T.

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Density Transformer:

For a density f on $[0,1]$, $\mathbf{H}[f]$ is the density on $[0,1]$ after one iteration of the shift

$$\mathbf{H}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|\, f \circ h(x) = \sum_{m \in \mathbb{N}} \frac{1}{(m+x)^2} f\left(\frac{1}{m+x}\right).$$

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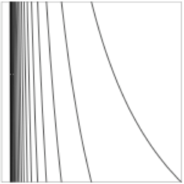

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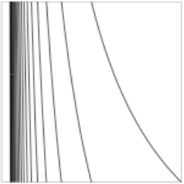

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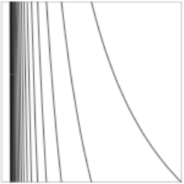

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The k-th iterate satisfies, with d extended in an additive way

$$\mathbf{H}^k_{s,w}[f](x) = \sum_{h\in\mathcal{H}^k} |h'(x)|^s\; e^{wd(h)} f\circ h(x)$$

II- Distributional results for weighted trajectories

Transfer operator and distributional study of weighted trajectories

In distributional studies, the main tools are the characteristic functions

$$\mathbb{E}[\exp(wD_n)], \qquad \mathbb{E}_N[\exp(w\widehat{D})]$$

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Real case: $\quad \mathbb{E}[\exp(wD_n)] = \int_{\mathcal{I}} \mathbf{H}_{1,w}^n[1](t)dt$

Rational case : $\quad \mathbb{E}_N[\exp(w\widehat{D})]$ related to $\quad [N^{-s}](I - \mathbf{H}_{s,w})^{-1}[1](0)$

due to the relation between

Dirichlet generating functions and quasi-inverses of the transfer operator,

$$S_d(s,w) := \sum_{(u,v)\in\Omega} \frac{1}{v^{2s}} \exp[w\widehat{D}(u,v)] = (I - \mathbf{H}_{s,w})^{-1}[1](0)$$

Distributional results for the continued fraction expansion

Already known results [Baladi-V (2003)]

In both cases, Real trajectories or Rational trajectories,

For a cost d of moderate growth $d(m) = O(\log m)$,

(a) Central Limit Theorems hold for $D_n, \widehat{D}_N$

(b) Moreover, for a lattice cost, Local Limit Theorems hold for $D_n, \widehat{D}_N$

$$\exists d_0, L \in \mathbb{R}, \quad \text{with } L > 0, \text{ such that} \quad \forall m \quad \frac{d(m) - d_0}{L} \in \mathbb{Z}$$

(c) With optimal speed of convergence

$$O\left(\frac{1}{\sqrt{n}}\right), \quad O\left(\frac{1}{\sqrt{\log N}}\right)$$

Distributional results for the continued fraction expansion

They deal with the characteristic functions $\mathbb{E}[\exp(wD_n)], \mathbb{E}_N[\exp(w\widehat{D})]$
and thus with the transfer operator $\mathbf{H}_{s,w}$

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For parameter s

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For parameter w:

- Central Limit Theorems:

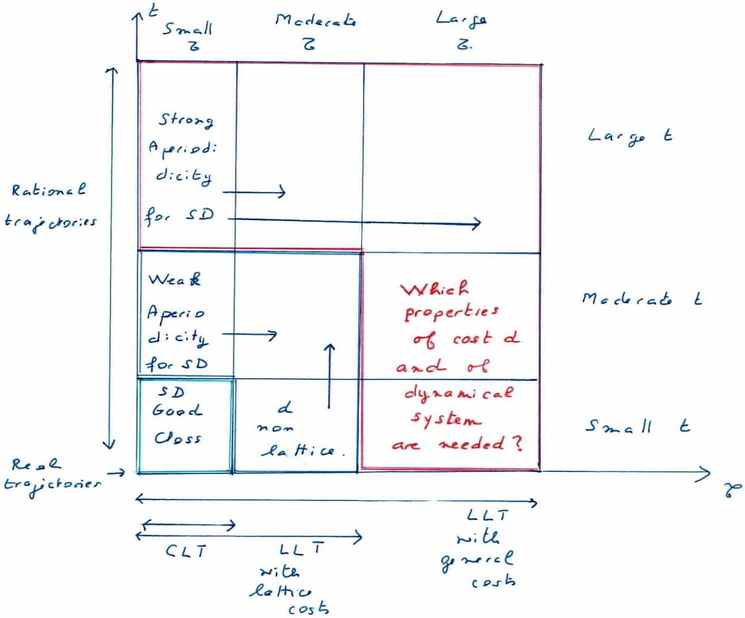
  $w \sim 0$
- Local Limit Theorems for a lattice cost :

  $w = i\tau$ with $\tau \in K$ compact $\subset \mathbb{R}$
- Local Limit Theorems for a non lattice cost :

  $w = i\tau$ with $\tau \in \mathbb{R}$

Properties of the dynamical system and cost needed in distributional studies

for dealing with the operator $\mathbf{H}_{1+it,i\tau}$ in each each domain (t,τ).



III- Local limit theorems with speed of convergence in simpler cases

Memoryless case.

Let (X_i) be a i.i.d sequence with values in $\mathbb{N}$, and $p_m := \Pr[X_i = m]$.
A cost $d : \mathbb{N} \to \mathbb{R}^+$, Some technical conditions:

$$\sigma_0 := \inf\{\sigma;\ \sum_{i=1}^{\infty} p_m^{\sigma} < \infty\} < 1, \qquad d(m) = O(|\log p_m|)$$

The mean $\mu[d]$ and the standard deviation $\sigma[d]$ exist. We assume $\sigma[d] \neq 0$.

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Main subject of interest: $\qquad D_n := \sum_{i=1}^{n} d(X_i) \qquad (n \to \infty).$

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There is a Central Limit Theorem (CLT) for D_n
with a speed of convergence of order $O(1/\sqrt{n})$,

$$\Pr\left[\frac{D_n - n\mu[d]}{\sigma[d]\sqrt{n}} \leq y\right] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt = O\left(\frac{1}{\sqrt{n}}\right).$$

A Local Limit Theorem (LLT)

- deals with $Q(x, n) := \mu[d]n + \delta[d]x\sqrt{n}$,
- evaluates the probability that $D_n - Q(x, n)$ belongs to some $J \subset \mathbb{R}$,
- compares it to $(|J|/\sqrt{2\pi n})\, e^{-x^2/2}$.

A Local Limit Theorem (LLT) proves that

$$\sqrt{n}\ \Pr[D_n - \mathbf{Q}(x, n) \in J] - |J| \frac{e^{-x^2/2}}{\delta(d)\sqrt{2\pi}} \to 0 \qquad (n \to \infty).$$

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What about the speed of convergence?

It depends on arithmetical properties of cost d. Two main cases:

the lattice case, and the non–lattice case.

A cost d is lattice if

$$\exists d_0, L \in \mathbb{R}, \quad \text{with } L > 0, \text{ such that } \quad \forall m \quad \frac{d(m) - d_0}{L} \in \mathbb{Z}$$

The smallest possible $L > 0$ is called the span of the lattice cost.

If $d_0 = 0$, the cost is called “plain lattice”.

In the lattice case, the optimal speed, of order $O(1/\sqrt{n})$ is attained.
More precisely, for a plain lattice cost of span 1, one has

$$\sqrt{n}\,\Pr[D_n = P(x,n)] = \sqrt{2\pi}\frac{e^{-x^2/2}}{\delta(d)} + O\left(\frac{1}{\sqrt{n}}\right) \qquad P(x,n) := \lfloor Q(x,n) \rfloor.$$

In this case, the characteristic function ϕ is periodic

$$\phi(\tau) := \int_{\mathbb{R}} \exp[i\tau x]\, dP_d(x) = \sum_{m \geq 1} p_m \exp[i\tau d(m)],$$

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Important fact: There is a relation between these two properties.

Proposition (classical and easy). The conditions are equivalent :

(i) The cost d is lattice

(ii) There exists $\tau_0 \neq 0$ for which ϕ_d satisfies $|\phi_d(\tau_0)| = 1$.

Moreover, Condition (ii) entails Condition (iii)

(iii) For any $h, k, \ell \in \mathbb{N}$, the ratio $\dfrac{d(h) - d(k)}{d(h) - d(\ell)}$ is rational.

Reinforcements of negations of Conditions (ii) or (iii).

A cost d is of characteristic exponent χ if

$$\exists K, \tau_0 > 0, \qquad |\phi_d(\tau)| \leq 1 - \frac{K}{|\tau|^\chi} \qquad \text{for } |\tau| \geq \tau_0.$$

A cost d is of diophantine exponent μ if

$$\exists (h, k, \ell) \in \mathbb{N}^3, \quad \text{such that the ratio} \quad \frac{d(h) - d(k)}{d(h) - d(\ell)} \quad \text{is } \mathtt{Diop}\ (\mu)$$

A number x is diophantine of exponent μ if

$$\exists C > 0, \quad \forall (p, q) \in \mathbb{N}^2, \quad \text{one has:} \qquad \left| x - \frac{p}{q} \right| > \frac{C}{q^{2+\mu}}$$

First result (Breuillard)

The cost d is of characteristic exponent χ $\Longrightarrow$ a Local Limit Theorem for D_n with speed $n^{1/\chi}$

For any ϵ with $\epsilon < 1/\chi$, for any compact interval $J \subset \mathbb{R}$,

there exists M_J, so that $\forall x \in \mathbb{R}, \forall n \geq 1$, one has:

$$\left| \sqrt{n}\ \Pr[D_n(u) - \mathbf{Q}(x,n) \in J] - |J| \frac{e^{-x^2/2}}{\delta(d)\sqrt{2\pi}} \right| \leq \frac{M_J}{n^\epsilon}$$

Second result (Breuillard)

The cost d is of diophantine exponent μ, $\Longrightarrow$ d of characteristic exponent χ for any $\chi > 2(\mu+1)$.

Conclusion:

The cost d is of diophantine exponent μ, $\Longrightarrow$ a Local Limit Theorem for D_n with speed $n^{1/2(\mu+1)}$.

IV- Local limit theorems with speed of convergence

Trajectories of dynamical systems.

And now if the X_i are generated by a dynamical system?

For instance the digits of the continued fraction expansion

(they are no longer independent)

Case of real trajectories

Definition: d is of characteristic exponent χ (wrt to the DS), if,

$$||\mathbf{H}_{1,i\tau}^{n(\tau)}|| \leq 1 - \frac{1}{|\tau|^{\chi}}, \qquad \text{for any } \tau \text{ with } |\tau| \geq \tau_0 \qquad n(\tau) := \Theta(\log|\tau]).$$

Two properties:

The cost d is of characteristic exponent χ wrt to the DS

$\Longrightarrow$ a Local Limit Theorem for D_n with speed $n^{1/\chi}$

The cost d is of diophantine exponent μ,

$\Longrightarrow$ d of characteristic exponent χ for any χ with $\chi > K(\mu + 1)$.

K depends on the DS.

A good generalization of the memoryless case.

Case of rational trajectories.

Definition: d is of uniform characteristic exponent χ

$$||\mathbf{H}_{1+it,i\tau}^{n(\tau)}|| \leq 1 - \frac{1}{|\tau|^{\chi}}, \qquad \text{for any } (t,\tau) \text{ with } |t| \leq a \text{ and } |\tau| \geq \tau_0.$$

NOW: (Baladi-Hachemi)

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Baladi and Hachemi proposed an intertwined diophantine condition

involving the branches of the dynamical system AND the cost d

Our result:

A set of two conditions NOT intertwined

- The diophantine condition (D) on the cost d
- A (new) condition (C) on the branches of the DS

  a "diophantine" version of the aperiodicity condition on the DS.

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The Aperiodicity Condition says :

"The branches of the system do not have all the same shape".

If $h^\star$ is the fixed point of branch h,

This implies that the cost $c(h) := \log |h'(h^\star)|$ is strongly non additive,

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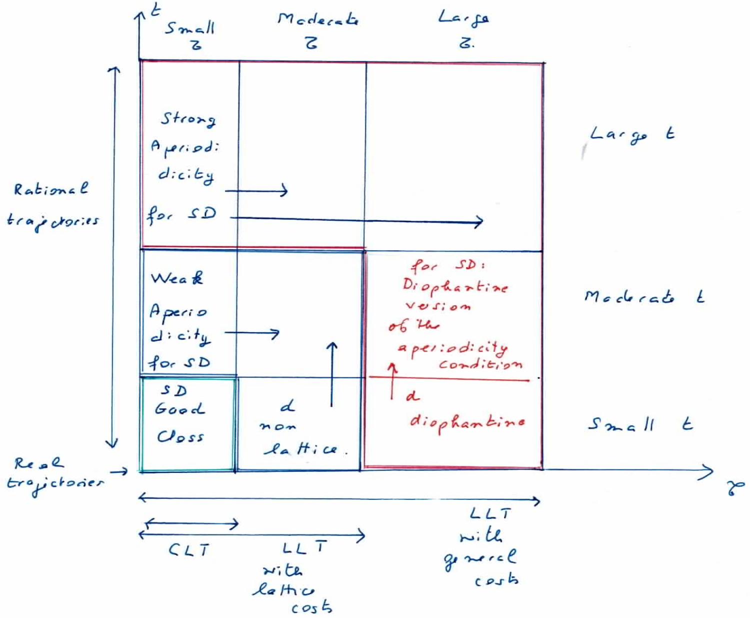
Our condition (C). There exist three branches h, k, ℓ for which

$$\Gamma(h,k) \neq 0, \quad \Gamma(h,\ell) \neq 0, \quad \text{and} \quad \frac{\Gamma(h,k)}{\Gamma(h,\ell)} \quad \text{is diophantine.}$$

Properties of the dynamical system and cost needed in distributional studies

for dealing with the operator $\mathbf{H}_{1+it,i\tau}$ in each each domain (t,τ).



Return to the Euclid dynamical system.

In this case, the condition (C) is always satisfied.

Let $c(h) := \log |h'(h^\star)|$. There exist three branches h, k, ℓ for which $$\Gamma(h,k) \neq 0, \quad \Gamma(h,\ell) \neq 0, \quad \text{and} \quad \frac{\Gamma(h,k)}{\Gamma(h,\ell)} \quad \text{is diophantine.}$$

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$\alpha(h,k)$ and $\alpha(h,\ell)$ be algebraically independent.

– Baker's theorem proves that the ratio $\Gamma(h,k)/\Gamma(h,\ell)$ is diophantine.

The final result,

for the total costs of a continued fraction relative to some cost d.

$$\widehat{D}_N(x) := \sum_{i=1}^{P(x)} d(m_i(x)) \qquad \text{on} \quad \Omega_N := \{x = p/q; \quad q \le N\}$$

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